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A SERIES SOLUTION OF THE EXACT EQUATION FOR THICK ORTHOTROPIC PLATES

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Abstract—By considering three-dimensional elasticity without any initial assumptions, the authors obtain the state equations for an orthotropic body. A series solution for a simply supported rectangular thick plate with arbitrary ratio between thickness and width under any given load is presented. Every fundamental equation of three-dimensional elasticity can be exactly satisfied and all the nine elastic constants can also be taken into account by the present method. Numerical results are obtained and compared with those of Reissner's and Ambartsumyan's theories and some references.

INTRODUCTION

The governing equation of thick plates which was first established by Reissner (1945) is based on the assumptions that the additional component $p(z)$ of displacements u and v and compressive deformation $B(z)$ are cubic functions across the thickness. After Reissner's work Hencky (1947) and Vlasov (1957) assumed that $p(z)$ is a linear and a cubic function, respectively. Mindlin (1951) proposed an improved theory by assuming $p(z)$ to be a sine function. However, the effect of the compressive deformation $B(z)$ is neglected in the above theories.

Among the theories of thick plates, Ambartsumyan's theory (Ambartsumyan, 1969), which introduces a given quadratic function to account for the variation of shearing stress across the thickness, seems to be more attractive to scholars.

All these theories are similar in that the deflection w has no relationship to coordinate z. The relation, however, has been considered by some higher-order theories, e.g. Lo *et al.* (1977).

With no initial assumptions regarding stress and deformation models, Vlasov proposed the Method of Initial Function (MIF) (Vlasov, 1957). Sundara Raja Iyengar applied this method to investigate the bending problem of a rectangular isotropic plate (Sundara Raja Iyengar *et al.*, 1974). Bahar, Das and Rao introduced the state space and matrix method to the MIF (Bahar, 1975; Das and Rao, 1977). For an isotropic body, the solutions of the initial functions can be obtained in a closed form by using the Cayley–Hamilton theorem. However, because the final equation is a transcendental one and finite terms of the Maclaurin series have to be used to solve it, the closed form is only a theoretical formulation. For an anisotropic body, it is very difficult to obtain the eigenvalues of a matrix with differential operators. Therefore, the solutions of the initial functions have to be expressed in the form of a Maclaurin series. Taking several terms of the series, all the physical quantities, in fact, appear to be polynomials of z in the solving process (Sundara Raja Iyengar and Pandya, 1983).

Considering three-dimensional elasticity and without any initial assumptions, the authors obtain the state equation for an orthotropic body. According to the boundary conditions of simply supported rectangular thick plates, all the physical quantities can be solved directly from the state equation. It is not necessary to deal with infinite-order partial differential equations since the state equation has not been expressed in the form of a Maclaurin series. Furthermore, the inconsistency among equations due to the cutting-error is avoided. The exact solutions for the flexure of static and dynamic plates with arbitrary elastic constants and the ratio between the thickness and width can easily be obtained.

FORMULATION OF THE STATE EQUATION

For an orthotropic body whose principal elastic directions coincide with the coordinate axes, let $X = \tau_{xz}$, $Y = \tau_{yz}$, $Z = \sigma_z$; U, V and W are the displacements along the x-, y- and z-directions, respectively; $\alpha = \partial/\partial x$, $\beta = \partial/\partial y$, $\xi^2 = \rho\,\partial^2/\partial t^2$; and ρ is the density of the material. By eliminating plane stresses σ_x, σ_y and τ_{xy} from the equilibrium, geometric and constitutive equations, the following state equation can be obtained:

$$\frac{\partial}{\partial Z}\begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & a_{11} & 0 & -\alpha \\ 0 & 0 & 0 & 0 & a_{22} & -\beta \\ 0 & 0 & 0 & -\alpha & -\beta & \xi^2 \\ \xi^2 - C_2\alpha^2 - C_6\beta^2 & -(C_3+C_6)\alpha\beta & C_1\alpha & 0 & 0 & 0 \\ -(C_3+C_6)\alpha\beta & \xi^2 - C_6\alpha^2 - C_4\beta^2 & C_5\beta & 0 & 0 & 0 \\ C_1\alpha & C_5\beta & C_{10} & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U \\ V \\ Z \\ X \\ Y \\ W \end{Bmatrix} \tag{1}$$

where

$$C_1 = -C_{13}/C_{33}, \quad C_2 = C_{11} - C_{13}^2/C_{33}, \quad C_3 = C_{12} - C_{13}C_{23}/C_{33},$$
$$C_4 = C_{22} - C_{23}^2/C_{33}, \quad C_5 = -C_{23}/C_{33}, \quad C_6 = C_{66},$$
$$C_{10} = 1/C_{33}, \quad a_{11} = 1/C_{55}, \quad a_{22} = 1/C_{44}.$$

C_{11}, C_{12}, C_{22}, ... are coefficients of rigidity which satisfy the following equalities:

$$C_{11} = E_x(1-\mu_{yz}\mu_{zy})/Q, \quad C_{22} = E_y(1-\mu_{xz}\mu_{zx})/Q, \quad C_{33} = E_z(1-\mu_{xy}\mu_{yx})/Q$$
$$C_{12} = E_x(\mu_{yx}+\mu_{zx}\mu_{zx}\mu_{yz})/Q, \quad C_{13} = E_x(\mu_{zx}+\mu_{yx}\mu_{zy})/Q,$$
$$C_{23} = E_y(\mu_{zy}+\mu_{xy}\mu_{zx})/Q$$
$$Q = 1-\mu_{xy}\mu_{yx}-\mu_{yz}\mu_{zy}-\mu_{zx}\mu_{xz}-2\mu_{xy}\mu_{yz}\mu_{zx}$$
$$C_{44} = G_{yz}, \quad C_{55} = G_{zx}, \quad C_{66} = G_{xy}, \quad E_x\mu_{yx} = E_y\mu_{xy},$$
$$E_x\mu_{zx} = E_z\mu_{zz}, \quad E_y\mu_{zy} = E_z\mu_{yz}.$$

The subscripts of E and G are the Young's modulus and shear modulus at given directions. μ_{xy} is Poisson's ratio which characterizes the contraction (expansion) in the direction of the y-axis during tension (compression) in the direction of the x-axis, and so forth.

By simple calculations, the eliminated stress components can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_2\alpha & C_3\beta & -C_1 \\ C_3\alpha & C_4\beta & -C_5 \\ C_6\beta & C_6\alpha & 0 \end{bmatrix}\begin{Bmatrix} U \\ V \\ Z \end{Bmatrix}. \tag{2}$$

SIMPLY SUPPORTED RECTANGULAR PLATE

The boundary conditions in this case are (Fig. 1):

$$\sigma_x = V = W = 0 \quad \text{at } x = 0, a$$
$$\sigma_y = U = W = 0 \quad \text{at } y = 0, b. \tag{3}$$

For static problems ($\xi = 0$), let

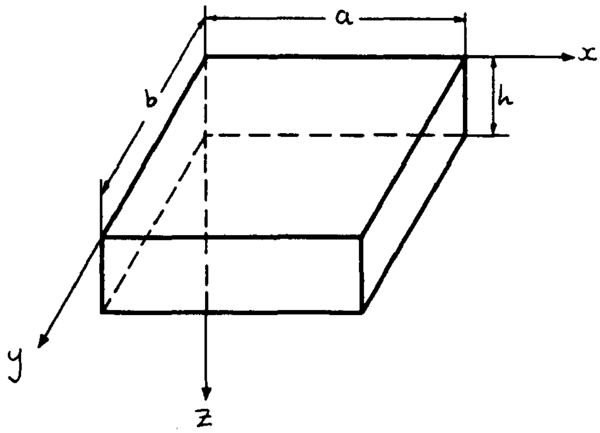

Fig. 1. Coordinate system and dimensions.

$$U = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(z) \cos(m\pi x/a) \sin(n\pi y/b)$$

$$V = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn}(z) \sin(m\pi x/a) \cos(n\pi y/b)$$

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(z) \sin(m\pi x/a) \sin(n\pi y/b). \quad (4)$$

Substitution of eqn (4) into eqn (1) gives

$$X = \sum_m \sum_n X_{mn}(z) \cos(m\pi x/a) \sin(n\pi y/b)$$

$$Y = \sum_m \sum_n Y_{mn}(z) \sin(m\pi x/a) \cos(n\pi y/b)$$

$$Z = \sum_m \sum_n Z_{mn}(z) \sin(m\pi x/a) \sin(n\pi y/b). \quad (5)$$

From eqn (2), we have

$$\sigma_x = \sum_m \sum_n \sigma_{x_{mn}}(z) \sin(m\pi x/a) \sin(n\pi y/b)$$

$$\sigma_y = \sum_m \sum_n \sigma_{y_{mn}}(z) \sin(m\pi x/a) \sin(n\pi y/b)$$

$$\tau_{xy} = \sum_m \sum_n \tau_{xy_{mn}}(z) \cos(m\pi x/a) \cos(n\pi y/b). \quad (6)$$

According to eqns (4) and (6), the boundary conditions (3) are satisfied.

Substitution of eqns (4) and (5) into eqn (1) gives the following expression for every m and n:

$$\frac{d}{dz}[U_{mn}(z) \quad V_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad W_{mn}(z)]^T =$$

$$\begin{bmatrix} 0 & A_{mn} \\ B_{mn} & 0 \end{bmatrix} [U_{mn}(z) \quad V_{mn}(z) \quad Z_{mn}(z) \quad X_{mn}(z) \quad Y_{mn}(z) \quad W_{mn}(z)]^T \quad (7)$$

in which

$$A_{mn}=\begin{bmatrix}a_{11} & 0 & -\zeta\\ 0 & a_{22} & -\eta\\ \zeta & \eta & 0\end{bmatrix};\quad B_{mn}=\begin{bmatrix}C_2\zeta^2+C_6\eta^2 & (C_3+C_6)\zeta\eta & C_1\zeta\\ (C_3+C_6)\zeta\eta & C_6\zeta^2+C_4\eta^2 & C_5\eta\\ -C_1\zeta & -C_5\eta & C_{10}\end{bmatrix}$$

$$\zeta = m\pi/a;\quad \eta = n\pi/b.$$

The solution for a set of differential equations (7) of first order with constant coefficients is

$$[U_{mn}(z)\quad V_{mn}(z)\quad Z_{mn}(z)\quad X_{mn}(z)\quad Y_{mn}(z)\quad W_{mn}(z)]^{\mathrm{T}} =$$

$$\exp\left\{z\begin{bmatrix}0 & A_{mn}\\ B_{mn} & 0\end{bmatrix}\right\}[U_{mn}(0)\quad V_{mn}(0)\quad Z_{mn}(0)\quad X_{mn}(0)\quad Y_{mn}(0)\quad W_{mn}(0)]^{\mathrm{T}} \quad (8)$$

where $U_{mn}(0), V_{mn}(0), \ldots, W_{mn}(0)$, which are called the initial values, are the values of $U_{mn}(z), V_{mn}(z), \ldots, W_{mn}(z)$ at $z = 0$.

If λ is the eigenvalue of the coefficient matrix in eqn (7), then λ must satisfy following eigenequation

$$\lambda^6+A_0\lambda^4+B_0\lambda^2+C_0 = 0 \quad (9)$$

in which A_0, B_0 and C_0 can be determined from the coefficient matrix.

For simplicity, we assume that the λ_s ($s = 1, 2, \ldots, 6$) are different and let $A = A_{mn}$, $B = B_{mn}$. By the Cayley–Hamilton theorem, we obtain

$$\exp\left\{z\begin{bmatrix}0 & A\\ B & 0\end{bmatrix}\right\} =$$

$$\begin{bmatrix}\alpha_1(z)I+\alpha_3(z)AB+\alpha_5(z)ABAB & \alpha_2(z)A+\alpha_4(z)ABA+\alpha_6(z)ABABA\\ \alpha_2(z)B+\alpha_4(z)BAB+\alpha_6(z)BABAB & \alpha_1(z)I+\alpha_3(z)BA+\alpha_5(z)BABA\end{bmatrix} \quad (10)$$

$$\begin{Bmatrix}\alpha_1(z)\\ \alpha_2(z)\\ \vdots\\ \alpha_6(z)\end{Bmatrix} = \begin{bmatrix}1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^5\\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^5\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & \lambda_6 & \lambda_6^2 & \ldots & \lambda_6^5\end{bmatrix}^{-1}\begin{Bmatrix}e^{\lambda_1 z}\\ e^{\lambda_2 z}\\ \vdots\\ e^{\lambda_6 z}\end{Bmatrix} \quad (11)$$

where I is a unit matrix. Substituting eqn (10) into eqn (8), we obtain all the physical quantities across the thickness:

$$[U_{mn}(z)\quad V_{mn}(z)\quad Z_{mn}(z)]^{\mathrm{T}} = [\alpha_1(z)I+\alpha_3(z)AB+\alpha_5(z)ABAB][U_{mn}(0)\quad V_{mn}(0)\quad Z_{mn}(0)]^{\mathrm{T}}$$
$$+[\alpha_2(z)A+\alpha_4(z)ABA+\alpha_6(z)ABABA][X_{mn}(0)\quad Y_{mn}(0)\quad W_{mn}(0)]^{\mathrm{T}}$$
$$[X_{mn}(z)\quad Y_{mn}(z)\quad W_{mn}(z)]^{\mathrm{T}} = [\alpha_1(z)I+\alpha_3(z)BA+\alpha_5(z)BABA][X_{mn}(0)\quad Y_{mn}(0)\quad W_{mn}(0)]^{\mathrm{T}}$$
$$+[\alpha_2(z)B+\alpha_4(z)BAB+\alpha_6(z)BABAB][U_{mn}(0)\quad V_{mn}(0)\quad Z_{mn}(0)]^{\mathrm{T}}. \quad (12)$$

The initial values on the right-hand side of the above equation can be determined according to the force vectors applied to the external plate planes ($z = 0, h$). For example, if the upper surface of a plate is subjected to arbitrary normal loads $q(x, y)$ and three concentrated forces p along the x-, y- and z-directions, respectively, at a given point (x_1, y_1), the boundary conditions can be expressed by:

$$\begin{Bmatrix} X_{mn}(0) \\ Y_{mn}(0) \\ Z_{mn}(0) \end{Bmatrix} =$$

$$\begin{Bmatrix} \frac{4p}{ab}\cos(m\pi x_1/a)\sin(n\pi y_1/b) \\ \frac{4p}{ab}\sin(m\pi x_1/a)\cos(n\pi y_1/b) \\ \frac{4p}{ab}\sin(m\pi x_1/a)\sin(n\pi y_1/b)+\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin(m\pi x/a)\sin(n\pi y/b)\,\mathrm{d}x\,\mathrm{d}y \end{Bmatrix}$$

$$[X_{mn}(h) \quad Y_{mn}(h) \quad Z_{mn}(h)]^{\mathrm{T}} = [0 \quad 0 \quad 0]^{\mathrm{T}}.$$

From eqn (11), one obtains

$$\alpha_s(z) = \sum_{j=1}^{6} d_{sj}\exp(\lambda_j z) = \sum_{j=1}^{6} d_{sj}K_j(z) \tag{13}$$

where $[d_{sj}]$ is the inverse matrix of the Vandermonde matrix in eqn (11). $K_j(z) = \exp(\lambda_j z)$.

After determining $\alpha_s(z)$ and the initial values, $U_{mn}(z)$, $V_{mn}(z), \ldots, W_{mn}(z)$ can be obtained from eqn (12). Then, we arrive at $\sigma_{x_{mn}}(z)$, $\sigma_{y_{mn}}(z)$ and $\tau_{xy_{mn}}(z)$ from eqn (2). Substitution of above values into eqns (4), (5) and (6) gives all the physical quantities.

NUMERICAL RESULTS

Example

A rectangular orthotropic thick plate with simply supported edges is subjected to uniform loads q_0 on its upper surface ($z = 0$). Numerical calculations were performed with

Table 1. Variations of maximum displacement components across thickness

z/h	Theories	$UE_z/(q_0h)$ $x = 0$, $y = b/2$	$VE_z/(q_0h)$ $x = a/2$, $y = 0$	$WE_z/(q_0h)$ $x = a/2$, $y = b/2$
0.0	Sundara	2.90269	4.37132	13.96010
0.2	Raja	1.40479	2.28731	13.94834
0.4	Iyengar	0.37976	0.52277	13.88676
0.6	and	−0.47322	−1.09933	13.79994
0.8	Pandya	−1.45498	−2.75600	13.68790
1.0		−2.86635	−4.62427	13.02603
0.0	Ambartsumyan	2.93512	4.45923	13.87199
0.2		1.38440	2.48948	13.87199
0.4		0.39869	0.79882	13.87199
0.6		−0.39869	−0.79882	13.87199
0.8		−1.38440	−2.48948	13.87199
1.0		−2.93512	−4.45923	13.87199
0.0	Reissner	2.48284	4.11641	13.57783
0.2		1.48971	2.46984	13.57783
0.4		0.49657	0.82428	13.57783
0.6		−0.49657	−0.82428	13.57783
0.8		−1.48971	−2.46984	13.57783
1.0		−2.48284	−4.11641	13.57783
0.0	Present study	2.89730	4.24230	13.72005
0.2		1.34129	2.16232	13.68747
0.4		0.36579	0.50036	13.60981
0.6		−0.45267	−1.04212	13.51160
0.8		−1.39751	−2.63049	13.39324
1.0		−2.80012	−4.42032	13.22745

Table 2. Variations of maximum stress components across thickness

z/h	Theories	δ_x/q_0 $x = a/2$, $y = b/2$	δ_y/q_0 $x = a/2$, $y = b/2$	δ_z/q_0 $x = a/2$, $y = b/2$	τ_{xy}/q_0 $x = 0$, $y = 0$	τ_{xz}/q_0 $x = 0$, $y = b/2$	τ_{yz}/q_0 $x = a/2$, $y = 0$
0.0	Sundara	−17.42338	−2.50500		3.96173	0.00000	0.00000
0.2	Raja	−9.07828	−1.50236		1.67037	2.32517	0.94468
0.4	Iyengar	−2.80469	−0.52956		0.16293	3.29399	1.38776
0.6	and	2.74531	0.43772		−0.99050	3.29399	1.38776
0.8	Pandya	8.91966	1.42751		−2.21985	2.32517	0.94468
1.0		17.06627	2.44285		−3.95503	0.00000	0.00000
0.0		−17.18955	−2.35704		3.53957	0.00000	0.00000
0.2		−8.85563	−1.36533		1.57267	2.26008	0.93027
0.4		−2.70886	−0.44696		0.43243	3.39001	1.39540
0.6	Ambartsumyan	2.70886	0.44690		−0.43243	3.39001	1.39540
0.8		8.85563	1.36533		−1.57267	2.26008	0.93027
1.0		17.18955	2.35704		−3.53957	0.00000	0.00000
0.0		−15.58077	−2.43819		2.68476	0.00000	0.00000
0.2		−9.34846	−1.46292		1.61088	2.24965	0.94000
0.4		−3.11615	−0.48764		0.53696	3.37447	1.41000
0.6	Reissner	3.11615	0.48764		−0.53696	3.37447	1.41000
0.8		9.34846	1.46292		−1.61088	2.24965	0.94000
1.0		15.58077	2.43819		−2.68476	0.00000	0.00000
0.0		−17.19845	−2.53812	−1.04284	2.90812	0.00000	0.00000
0.2		−8.89123	−1.49684	−0.90046	1.40428	2.53263	1.10795
0.4		−2.74270	−0.52559	−0.65114	0.31457	3.17441	1.34102
0.6	Present study	2.68282	0.43337	−0.35499	−0.63132	3.04427	1.23616
0.8		8.77416	1.40553	−0.10508	−1.64821	2.12289	0.81539
1.0		16.96432	2.41746	0.00000	−2.96767	0.00000	0.00000

the following values (Fig. 1):

$$E_x = 10E_y = 10E_z, \quad G_{xy} = G_{xz} = 0.6E_z, \quad G_{yz} = 0.5E_z$$

$$\mu_{xy} = \mu_{xz} = \mu_{yz} = 0.25, \quad a = b, \quad h/a = 0.2.$$

The results can be found in Tables 1 and 2. All the calculations were carried out on a VAX-11/780 with double precision whereby 15 terms in each of the x and y variables were retained for each Fourier series ($m, n = 1, 3, 5, \ldots, 29$).

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